

# On the Equivalence of Holographic and Complex Embeddings for Link Prediction

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## Abstract

We show the equivalence of two state-of-the-art link prediction/knowledge graph completion methods: Nickel et al.'s holographic embedding and Trouillon et al.'s complex embedding. We first consider a spectral version of the holographic embedding, exploiting the frequency domain in the Fourier transform for efficient computation. The analysis of the resulting method reveals that it can be viewed as an instance of the complex embedding with certain constraints cast on the initial vectors upon training. Conversely, any complex embedding can be converted to an equivalent holographic embedding.

## 1 Introduction

Knowledge graph completion (Nickel et al., 2015) aims at augmenting missing knowledge in an incomplete knowledge base. It can be viewed as a task of link prediction (Liben-Nowell and Kleinberg, 2003; Hasan and Zaki, 2011), and various approaches based on vector embedding have been proposed in recent years (Bordes et al., 2011; Socher et al., 2013; Guu et al., 2015; Yang et al., 2015; Nickel et al., 2016; Trouillon et al., 2016b). Holographic embedding (Nickel et al., 2016) is one of the state-of-the-art methods along this line of research.

In this paper, first we show that holographic embedding can be trained entirely in the frequency domain induced by the Fourier transform, thereby reducing the computation time of the scoring function from  $O(n \log n)$  to  $O(n)$ . The analysis of the resulting method reveals that holographic embedding and Trouillon et al. (2016b)'s complex embedding, which is another state-of-the-art method for knowledge graph completion, differ only in terms of the constraints on the initial values and how a real-valued score is obtained from complex-valued dot product.

We also show that every complex embedding has an equivalent holographic embedding (with real vectors) in the sense that their scoring functions are equal up to scaling.

## 2 Preliminaries

Let  $i$  denote the imaginary unit,  $\mathbb{R}$  be the set of real values, and  $\mathbb{C}$  the set of complex values. For a vector  $\mathbf{v}$ ,  $[\mathbf{v}]_j$  represents the  $j$ th component of  $\mathbf{v}$ . For a complex vector  $\mathbf{z}$ , and matrix  $\mathbf{Z}$ ,  $\mathbf{z}^T$  and  $\mathbf{Z}^T$  respectively denote the transpose of  $\mathbf{z}$  and  $\mathbf{Z}$ . For a complex scalar  $z$ , vector  $\mathbf{z}$ , and matrix  $\mathbf{Z}$ ,  $\bar{z}$ ,  $\bar{\mathbf{z}}$ , and  $\bar{\mathbf{Z}}$  are their complex conjugate.

Let  $\mathbf{x} = [x_0 \cdots x_{n-1}]^T \in \mathbb{R}^n$  and  $\mathbf{y} = [y_0 \cdots y_{n-1}]^T \in \mathbb{R}^n$ . Note that the vector indices start from 0 for notational convenience. The *circular convolution* of  $\mathbf{x}$  and  $\mathbf{y}$ , denoted by  $\mathbf{x} * \mathbf{y}$ , is defined by

$$[\mathbf{x} * \mathbf{y}]_j = \sum_{k=0}^{n-1} x_{[(j-k) \bmod n]} y_k. \quad (1)$$

Likewise, *circular correlation*  $\mathbf{x} \star \mathbf{y}$  is defined by

$$[\mathbf{x} \star \mathbf{y}]_j = \sum_{k=0}^{n-1} x_{[(k-j) \bmod n]} y_k. \quad (2)$$

While circular convolution is commutative, circular correlation is not; i.e.,  $\mathbf{x} * \mathbf{y} = \mathbf{y} * \mathbf{x}$ , but  $\mathbf{x} \star \mathbf{y} \neq \mathbf{y} \star \mathbf{x}$  in general. As it can be verified with Eqs. (1) and (2),  $\mathbf{x} \star \mathbf{y} = \text{flip}(\mathbf{x}) * \mathbf{y}$ , where  $\text{flip}(\mathbf{x}) = [x_{n-1} \cdots x_0]^T$  is a vector obtained by arranging the components of  $\mathbf{x}$  in reverse.

For  $n$ -dimensional vectors, naively computing circular convolution/correlation by Eqs. (1) and (2) requires  $O(n^2)$  multiplications. However, we can take advantage of the discrete Fourier transform (DFT) to accelerate the computation: For circular convolution, first compute the DFTs of  $\mathbf{x}$  and  $\mathbf{y}$ , and then compute the inverse DFT of their elementwise vector product, i.e.,

$$\mathbf{x} * \mathbf{y} = \mathfrak{F}^{-1}(\mathfrak{F}(\mathbf{x}) \odot \mathfrak{F}(\mathbf{y})),$$

where  $\mathfrak{F} : \mathbb{R}^n \rightarrow \mathbb{C}^n$  and  $\mathfrak{F}^{-1} : \mathbb{C}^n \rightarrow \mathbb{R}^n$  respectively denote the DFT and inverse DFT, and  $\odot$  denotes the elementwise product. Since DFT and inverse DFT can be computed in  $O(n \log n)$  time with the Fast Fourier Transform algorithm, the computation time for circular convolution is also  $O(n \log n)$ . The same can be said of circular correlation. Since  $\mathfrak{F}(\text{flip}(\mathbf{x})) = \overline{\mathfrak{F}(\mathbf{x})}$ , we have

$$\mathbf{x} \star \mathbf{y} = \mathfrak{F}^{-1}(\overline{\mathfrak{F}(\mathbf{x})} \odot \mathfrak{F}(\mathbf{y})). \quad (3)$$

In analogy to signal processing application of the Fourier transform, the original real space  $\mathbb{R}^n$  is customarily called the “time” domain, and the complex space  $\mathbb{C}^n$  where DFT vectors reside is called the “frequency” domain.

## 3 Holographic embedding for knowledge graph completion

### 3.1 Knowledge graph completion

Let  $\mathcal{E}$  and  $\mathcal{R}$  be finite sets of *entities* and (*binary*) *relations* over entities, respectively. For each relation  $r \in \mathcal{R}$  and each pair  $s, o \in \mathcal{E}$  of entities, we are interested in whether  $r(s, o)$

Table 1: Correspondence between operations in time and frequency domains.  $\mathbf{r} \leftrightarrow \mathbf{p}$  indicates  $\mathbf{p} = \mathfrak{F}(\mathbf{r})$  (and also  $\mathbf{r} = \mathfrak{F}^{-1}(\mathbf{p})$ ).

operation	time		frequency
scalar mult.	$\alpha \mathbf{x}$	$\longleftrightarrow$	$\alpha \mathfrak{F}(\mathbf{x})$
summation	$\mathbf{x} + \mathbf{y}$	$\longleftrightarrow$	$\mathfrak{F}(\mathbf{x}) + \mathfrak{F}(\mathbf{y})$
flip	$\text{flip}(\mathbf{x})$	$\longleftrightarrow$	$\overline{\mathfrak{F}(\mathbf{x})}$
convolution	$\mathbf{x} * \mathbf{y}$	$\longleftrightarrow$	$\mathfrak{F}(\mathbf{x}) \odot \mathfrak{F}(\mathbf{y})$
correlation	$\mathbf{x} \star \mathbf{y}$	$\longleftrightarrow$	$\overline{\mathfrak{F}(\mathbf{x})} \odot \mathfrak{F}(\mathbf{y})$
dot product	$\mathbf{x} \cdot \mathbf{y}$	$=$	$\frac{1}{n} \mathfrak{F}(\mathbf{x}) \cdot \mathfrak{F}(\mathbf{y})$

holds<sup>1</sup>; we write  $r(s, o) = +1$  if it holds, and  $r(s, o) = -1$  if not. To be precise, given a *training set*  $\mathcal{D} = \mathcal{R} \times \mathcal{E} \times \mathcal{E} \times \{-1, +1\}$  such that  $(r, s, o, y) \in \mathcal{D}$  indicates  $y = r(s, o)$ , we are to design a function  $f : \mathcal{R} \times \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R}$  such that  $f(r, s, o)$  gives an estimated score of  $r(s, o) = +1$  for triples  $(r, s, o)$  not observed in  $\mathcal{D}$ ; function  $f$  should give a higher value for triples  $(r, s, o)$  if  $r(s, o) = +1$  is more likely, and a smaller value for those that are less likely. If necessary,  $f(r, s, o)$  can be converted to probability by  $P[r(s, o) = +1] = \sigma(f(r, s, o))$ , where  $\sigma : \mathbb{R} \rightarrow (0, 1)$  is a sigmoid function.

Dataset  $\mathcal{D}$  can be regarded as a directed graph with nodes representing entities  $\mathcal{E}$  and edges labeled by relations  $\mathcal{R}$ . Thus, the task is basically that of *link prediction* (Liben-Nowell and Kleinberg, 2003; Hasan and Zaki, 2011). Often, it is also called *knowledge graph completion*.

### 3.2 Holographic embedding (HolE)

Nickel et al. (2016) proposed *holographic embedding* (HolE) for knowledge graph completion. Using training data  $\mathcal{D}$ , this method learns the vector embeddings  $\mathbf{e}_k \in \mathbb{R}^n$  of entities  $k \in \mathcal{E}$  and the embeddings  $\mathbf{w}_r \in \mathbb{R}^n$  of relations  $r \in \mathcal{R}$ . The score for triple  $(r, s, o)$  is then given by

$$f_{\text{HolE}}(r, s, o) = \mathbf{w}_r \cdot (\mathbf{e}_s \star \mathbf{e}_o). \quad (4)$$

Eq. (4) can be evaluated in time  $O(n \log n)$  if  $\mathbf{e}_s \star \mathbf{e}_o$  is computed by Eq. (3).

## 4 Spectral training of HolE

Nickel et al. (2016) used DFT and the resulting “frequency” vectors to compute circular correlation efficiently. In this section, we extend this technique further, and consider training HolE solely in the frequency domain. That is, real-valued embeddings  $\mathbf{e}_k, \mathbf{w}_r \in \mathbb{R}^n$  in the original “time” domain are abolished, and instead we train their DFT counterparts  $\boldsymbol{\epsilon}_k = \mathfrak{F}(\mathbf{e}_k) \in \mathbb{C}^n$  and  $\boldsymbol{\omega}_k = \mathfrak{F}(\mathbf{w}_r) \in \mathbb{C}^n$  in the frequency domain. This formulation eliminates the need of

<sup>1</sup>Depending on the context, letter  $r$  is used either as an index to an element in  $\mathcal{R}$  or the binary relation it signifies.

DFT/inverse DFT, which is the major computational bottleneck in HolE. In particular, the scoring function of Eq. (4) can be computed in  $O(n)$  time directly from  $\mathbf{\epsilon}_k$  and  $\mathbf{\omega}_k$ .

Indeed, equivalent counterparts in the frequency domain exist for not only convolution/correlation but all other computations needed for HolE: scalar multiplication, summation (needed when vectors are updated with stochastic gradient descent), and dot product (used in Eq. (4)). The frequency-domain equivalents for these operations are summarized in Table 1. All of these can be performed efficiently (in linear time) in the frequency domain.

Concerning dot product, the following relation holds for  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ :

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{n} \mathfrak{F}(\mathbf{x}) \cdot \mathfrak{F}(\mathbf{y}). \quad (5)$$

This relation is known as Parseval’s theorem (also called the *power theorem* (Smith, 2007)), and it states that dot products in two domains are equal up to scaling.

After embeddings  $\mathbf{\epsilon}_k, \mathbf{\omega}_r \in \mathbb{C}^n$  are learned in the frequency domain, their time-domain counterparts  $\mathbf{e}_k = \mathfrak{F}^{-1}(\mathbf{\epsilon}_k)$  and  $\mathbf{w}_r = \mathfrak{F}^{-1}(\mathbf{\omega}_r)$  can be recovered if needed, but this is not required as far as computation of the scoring function is concerned. Using Parseval’s theorem, Eq. (4) can be directly computed from the frequency vectors  $\mathbf{\epsilon}_k, \mathbf{\omega}_r \in \mathbb{C}^n$  by

$$f_{\text{HolE}}(r, s, o) = \frac{1}{n} \mathbf{\omega}_r \cdot (\overline{\mathbf{\epsilon}_s} \odot \mathbf{\epsilon}_o). \quad (6)$$

#### 4.1 Conjugate symmetry of spectral components

A complex vector  $\boldsymbol{\xi} = [\xi_0 \cdots \xi_{n-1}]^T \in \mathbb{C}^n$  is said to be *conjugate symmetric* (or *Hermitian*) if  $\xi_j = \overline{\xi_{[(n-j) \bmod n]}}$  for  $j = 0, \dots, n-1$ , or, in other words, if it can be written in the form

$$\boldsymbol{\xi} = \begin{cases} \begin{bmatrix} \xi_0 & \boldsymbol{\Upsilon} & \text{flip}(\overline{\boldsymbol{\Upsilon}}) \end{bmatrix}^T, & \text{if } n \text{ is odd,} \\ \begin{bmatrix} \xi_0 & \boldsymbol{\Upsilon} & \xi_{n/2} & \text{flip}(\overline{\boldsymbol{\Upsilon}}) \end{bmatrix}^T, & \text{if } n \text{ is even,} \end{cases}$$

for some  $\boldsymbol{\Upsilon} \in \mathbb{C}^{[n/2]-1}$ , with  $\xi_0, \xi_{n/2} \in \mathbb{R}$ .

The DFT  $\mathfrak{F}(\mathbf{x})$  is conjugate symmetric if and only if  $\mathbf{x}$  is a real vector. Thus, maintaining conjugate symmetry of “frequency” vectors is the key to ensure their “time” counterparts remain in real space. Below, we verify that this property is indeed preserved with stochastic gradient descent. It also provides a sufficient condition under which dot product takes a real value, and is also relevant to the discussion of space requirement.

#### 4.2 Initialization and value update in frequency domain

Typically, at the beginning of training HolE, each individual embedding is initialized by a random vector. When we train HolE in the frequency domain, we could first generate a random real vector, regard them as a HolE vector in the time domain, and compute its DFT as the initial value in the frequency domain. An alternative, easier approach is to directly generate a random complex vector that is conjugate symmetric, and use it as the initial frequency vector. This guarantees the inverse DFT to be a real vector, i.e., there exists a valid corresponding image in the time domain.

During the stochastic gradient descent (SGD) training, vectors  $\omega_r, \epsilon_s, \epsilon_o$  are updated respectively by  $\alpha \nabla_{\omega_r} f, \alpha \nabla_{\epsilon_s} f, \alpha \nabla_{\epsilon_o} f$ , where  $\alpha \in \mathbb{R}$  is a factor not depending on these parameters, and

$$\begin{aligned}\nabla_{\omega_r} f &= \overline{\epsilon_s} \odot \epsilon_o, \\ \nabla_{\epsilon_s} f &= \overline{\omega_r} \odot \epsilon_o, \\ \nabla_{\epsilon_o} f &= \omega_r \odot \epsilon_s.\end{aligned}$$

As seen from above, conjugation, scalar multiplication, summation, and elementwise product are used in the SGD update. And it is easy to verify that all these operations preserve conjugate symmetry. It follows that if  $\omega_r, \epsilon_s, \epsilon_o$  are initially conjugate symmetric, they will remain so during the course of training, assuring the inverse DFT of the learned embeddings to be real vectors.

### 4.3 Real-valued dot product

In the scoring function of HolE (Eq. (4)), dot product is used for generating a real-valued “score” out of two vectors,  $\mathbf{w}_r$  and  $\mathbf{e}_s \odot \mathbf{e}_o$ . Likewise, in Eq. (6), the dot product is applied to  $\omega_r$  and  $\epsilon_s \odot \epsilon_o$ , which are complex-valued. However, provided that the conjugate symmetry of these vectors are maintained, their dot product is always real. This follows from Parseval’s theorem; the inverse DFTs of these frequency vectors are real, and thus their dot product is also real. Therefore, the dot product of the corresponding frequency vectors must be real, too, according to Eq. (5).

### 4.4 Space requirement

A general complex vector  $\xi \in \mathbb{C}^n$  can be stored in memory as  $2n$  floating-point numbers, i.e., one each for the real and imaginary part of a component. In our spectral representation of HolE, however, the first  $\lfloor n/2 \rfloor$  components suffice to specify the frequency vector  $\xi$ , since the vector is conjugate symmetric. Moreover,  $\xi_0$  (and  $\xi_{n/2}$  if  $n$  is even) are real values. Thus, a spectral representation of HolE can be specified with exactly  $n$  floating-point numbers, which can be stored in the same amount of memory as needed by the original HolE.

## 5 Relation to Trouillon et al.’s complex embedding

### 5.1 Complex embedding (CompE)

Trouillon et al. (2016b) proposed a method for embedding-based knowledge graph completion, called *complex embedding* (CompE). The objective is the same as Nickel et al.’s; the embedding  $\mathbf{e}_k$  of entities and  $\mathbf{w}_r$  of relations are to be learned. In their model, however, these vectors are complex-valued, and is based on the eigendecomposition of complex matrices  $\mathbf{X}_r = \mathbf{E} \mathbf{W}_r \overline{\mathbf{E}}^T$  that encodes relation  $r \in \mathcal{R}$  over pairs of entities, where  $\mathbf{X}_r \in \mathbb{C}^{|\mathcal{E}| \times |\mathcal{E}|}$ ,  $\mathbf{E} = [\mathbf{e}_1, \dots, \mathbf{e}_{|\mathcal{E}|}]^T \in \mathbb{C}^{|\mathcal{E}| \times n}$ , and  $\mathbf{W}_r = \text{diag}(\mathbf{w}_r) \in \mathbb{C}^{n \times n}$  is a diagonal matrix (with diagonal elements  $\mathbf{w}_r \in \mathbb{C}^n$ ). In practice,  $\mathbf{X}_r$  needs to be a real matrix, because its  $(r, s)$ -component defines the score for  $(r, s, o)$ . To this end, Trouillon et al. simply extracted the real part; i.e.,

$\mathbf{X}_r = \text{Re}(\mathbf{E}\mathbf{W}_r\bar{\mathbf{E}}^T)$ , where  $\text{Re}(\mathbf{Z})$  denotes the real part of complex-valued matrix  $\mathbf{Z}$ . Trouillon et al. (2016a) advocated this approach, by showing that any real matrix  $\mathbf{X}_r$  can be expressed in this form.

With this formulation, the score for triple  $(r, s, o)$  is given by

$$f_{\text{CompE}}(r, s, o) = \text{Re} \left( \sum_{j=0}^{n-1} [\mathbf{w}_r]_j [\mathbf{e}_s]_j \overline{[\mathbf{e}_o]_j} \right). \quad (7)$$

## 5.2 Equivalence of holographic and complex embeddings

Now let us rewrite Eq. (7). Noting the definition of complex dot product, i.e.,  $\mathbf{a} \cdot \mathbf{b} = \bar{\mathbf{a}}^T \mathbf{b}$ , we have

$$\begin{aligned} \sum_{j=0}^{n-1} [\mathbf{w}_r]_j [\mathbf{e}_s]_j \overline{[\mathbf{e}_o]_j} &= (\mathbf{e}_s \odot \bar{\mathbf{e}}_o)^T \mathbf{w}_r \\ &= \overline{(\mathbf{e}_s \odot \bar{\mathbf{e}}_o)} \cdot \mathbf{w}_r & (\because \mathbf{a} \cdot \mathbf{b} = \bar{\mathbf{a}}^T \mathbf{b}) \\ &= (\bar{\mathbf{e}}_s \odot \mathbf{e}_o) \cdot \mathbf{w}_r \\ &= \overline{\mathbf{w}_r \cdot (\bar{\mathbf{e}}_s \odot \mathbf{e}_o)} & (\because \mathbf{a} \cdot \mathbf{b} = \overline{\mathbf{b} \cdot \mathbf{a}}) \end{aligned}$$

and since  $\text{Re}(\mathbf{z}) = \text{Re}(\bar{\mathbf{z}})$ ,

$$\text{Re}(\overline{\mathbf{w}_r \cdot (\bar{\mathbf{e}}_s \odot \mathbf{e}_o)}) = \text{Re}(\mathbf{w}_r \cdot (\bar{\mathbf{e}}_s \odot \mathbf{e}_o)).$$

Thus, Eq. (7) can be written as

$$f_{\text{CompE}}(r, s, o) = \text{Re}(\mathbf{w}_r \cdot (\bar{\mathbf{e}}_s \odot \mathbf{e}_o)). \quad (8)$$

In Eq. (8), we immediately observe a striking similarity to Eq. (6), the scoring function for HolE. The complex embedding extracts the real part of complex dot product, whereas in our spectral version of HolE (spectral HolE), dot product is guaranteed to be real because all embeddings satisfy conjugate symmetry. Indeed, Eq. (6) can be equally written as

$$f_{\text{HolE}}(r, s, o) = \frac{1}{n} \text{Re}(\mathbf{w}_r \cdot (\bar{\mathbf{e}}_s \odot \mathbf{e}_o)).$$

The operator  $\text{Re}(\cdot)$  in this formula is redundant as the value is guaranteed to be real-valued, but this equation further elucidates the similarity between Eq. (6) and Eq. (8).

As argued in Section 4, the original HolE has an equivalent complex-valued embedding in spectral HolE, in the sense that their scoring functions always return the same score given the same triple. And as shown above, spectral HolE can be regarded as a special case of complex embedding trained with a specific form of initial vectors, i.e., conjugate symmetric vectors. It follows that a model that can be computed by HolE, either original or spectral, can also be computed by the complex embedding.

Conversely, given a complex embedding, we can construct an equivalent HolE, in the sense that  $f_{\text{CompE}}(r, s, o) = c f_{\text{HolE}}(r, s, o)$  for every  $r, s, o$ , where  $c > 0$  is a constant. For each

$n$ -dimensional complex embeddings  $\mathbf{x} \in \{\mathbf{e}_k\}_{k \in \mathcal{E}} \cup \{\mathbf{w}_r\}_{r \in \mathcal{R}} \subset \mathbb{C}^n$  computed by a CompE, we make a corresponding HolE  $\mathfrak{h}(\mathbf{x}) \in \mathbb{R}^{2n+1}$  as follows: For a given complex embedding  $\mathbf{x} = [x_0 \cdots x_{n-1}] \in \mathbb{C}^n$ , first compute  $\mathfrak{s}(\mathbf{x}) \in \mathbb{C}^{2n+1}$  by

$$\begin{aligned}\mathfrak{s}(\mathbf{x}) &= [0 \quad x_0 \quad \cdots \quad x_{n-1} \quad \overline{x_{n-1}} \quad \cdots \quad \overline{x_0}]^T \\ &= [0 \quad \mathbf{x} \quad \text{flip}(\overline{\mathbf{x}})]^T\end{aligned}\tag{9}$$

and then define  $\mathfrak{h}(\mathbf{x}) = \mathfrak{F}^{-1}(\mathfrak{s}(\mathbf{x}))$ .

Since  $\mathfrak{s}(\mathbf{x})$  is conjugate symmetric,  $\mathfrak{h}(\mathbf{x})$  is a real vector. This allows us to regard  $\mathfrak{h}(\mathbf{w}_r)$ ,  $\mathfrak{h}(\mathbf{e}_s)$ ,  $\mathfrak{h}(\mathbf{e}_o) \in \mathbb{R}^{2n+1}$  as holographic embeddings of  $r$ ,  $s$  and  $o$ , and compute the score:

$$\begin{aligned}f_{\text{HolE}}(r, s, o) &= \mathfrak{h}(\mathbf{w}_r) \cdot (\mathfrak{h}(\mathbf{e}_s) \star \mathfrak{h}(\mathbf{e}_o)) \\ &= \frac{1}{n} \mathfrak{s}(\mathbf{w}_r) \cdot (\overline{\mathfrak{s}(\mathbf{e}_s)} \odot \mathfrak{s}(\mathbf{e}_o)) && (\because \text{Eq. (6)}) \\ &= \frac{1}{n} \mathfrak{s}(\mathbf{w}_r) \cdot \left[ 0 \quad \overline{\mathbf{e}_s} \odot \mathbf{e}_o \quad \text{flip}(\overline{\mathbf{e}_s} \odot \mathbf{e}_o) \right]^T && (\because \text{Eq. (9)}) \\ &= \frac{1}{n} [0 \quad \mathbf{w}_r \quad \text{flip}(\overline{\mathbf{w}_r})]^T \cdot \left[ 0 \quad \overline{\mathbf{e}_s} \odot \mathbf{e}_o \quad \text{flip}(\overline{\mathbf{e}_s} \odot \mathbf{e}_o) \right]^T \\ &= \frac{1}{n} \left( \mathbf{w}_r \cdot (\overline{\mathbf{e}_s} \odot \mathbf{e}_o) + \text{flip}(\overline{\mathbf{w}_r}) \cdot \text{flip}(\overline{\mathbf{e}_s} \odot \mathbf{e}_o) \right) \\ &= \frac{1}{n} \left( \mathbf{w}_r \cdot (\overline{\mathbf{e}_s} \odot \mathbf{e}_o) + \overline{\mathbf{w}_r} \cdot \overline{(\mathbf{e}_s \odot \mathbf{e}_o)} \right) \\ &= \frac{1}{n} \left( \mathbf{w}_r \cdot (\overline{\mathbf{e}_s} \odot \mathbf{e}_o) + \overline{\mathbf{w}_r \cdot (\mathbf{e}_s \odot \mathbf{e}_o)} \right) \\ &= \frac{2}{n} \text{Re}(\mathbf{w}_r \cdot (\overline{\mathbf{e}_s} \odot \mathbf{e}_o)) \\ &= \frac{2}{n} f_{\text{CompE}}(r, s, o),\end{aligned}$$

which shows that  $\mathfrak{h}(\cdot)$  (or  $\mathfrak{s}(\cdot)$ ) gives the desired conversion from CompE to HolE.

## 6 Conclusion

We have shown that the holographic embedding (HolE) can be computed entirely in the frequency domain, thereby reducing the computation time of the scoring function from  $O(n \log n)$  to  $O(n)$ . In particular, we showed that the conjugate symmetry of frequency vectors is preserved by stochastic gradient descent, which ensures the existence of the corresponding holographic embedding in the original space (time domain) is guaranteed.

Also, we have established the equivalence of HolE and the complex embedding: The spectral version of HolE is subsumed by the complex embedding as a special case in which the conjugate symmetry is imposed on embeddings. Conversely, every complex embedding has an equivalent HolE.

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